Math 111-01, Fall 2023 Professor Rob Benedetto

Supplement to Worksheet 3

This handout presents some examples of derivative problems like those near the end of Worksheet 3, which involve the **limit definition of the derivative**. You should use these examples as **model solutions** for how to compute derivatives from the limit definition.

The Limit Definition of the Derivative

Let f be a function, and let a be a real number in the domain of f. Then the **derivative of f at $x = a$** is:

$$\boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}} \qquad \textbf{IF} \text{ the limit converges.}$$

The intuitive meaning of $f'(a)$, which you should read aloud as "f prime of a," is any or all of:

- the slope of the tangent line to the graph $y = f(x)$ at the point $x = a$
- the instantaneous velocity at time a, if $f(x)$ means the position of a moving object at time x
- the instantaneous rate of change of $f(x)$ with respect to x

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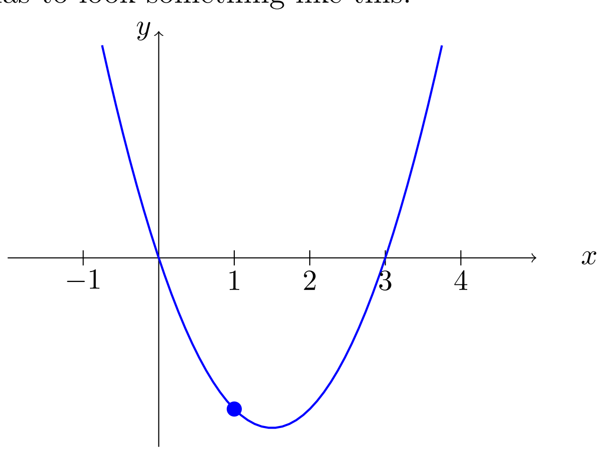
Example 1. Let $f(x) = x^2 - 3x$. Find the slope of the tangent line to the curve $y = f(x)$ at the point where $x = 1$.

Solution: The slope of the tangent line is the derivative:

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 3(1+h) - [1^2 - 3\cdot 1]}{h}$$

$$= \lim_{h\to 0} \frac{1 + 2h + h^2 - 3 - 3h - 1 + 3}{h} = \lim_{h\to 0} \frac{-h + h^2}{h} = \lim_{h\to 0}(-1 + h) \overset{\text{DSP}}{=} \boxed{-1}$$

FYI: although Example 1 didn't ask for a graph, here's one anyhow. (So what follows is *not* part of the model solution, but just some hopefully helpful commentary.) We know that $y = x^2 - 3x$ is a parabola opening upward, and writing it as $y = x(x-3)$, we can see it crosses the x-axis at $x = 0$ and $x = 3$. So the graph has to look something like this:



And if you try to eyeball the tangent line at the point I marked in — which is the point $(1, f(1))$, i.e., the point $(1, -2)$ — it's clear that it will have negative slope, and certainly plausible that that slope is, in fact, -1. The computation above shows that yes, its slope is *exactly* -1.

Example 2. A car is driving down a road with position at time t (hours) given by $s(t) = 60\sqrt{t+1}$ (km). What is the car's (instantaneous) velocity at time $t = 3$?
Solution: The velocity is the derivative of the position, so:

$$s'(3) = \lim_{h\to 0} \frac{s(3+h) - s(3)}{h} = \lim_{h\to 0} \frac{60\sqrt{3+h+1} - 60\sqrt{3+1}}{h} = \lim_{h\to 0} \frac{60(\sqrt{4+h} - 2)}{h}$$
$$= \lim_{h\to 0} \frac{60(\sqrt{4+h} - 2)(\sqrt{4+h} + 2)}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{60((4+h) - 4)}{h(\sqrt{4+h} + 2)}$$
$$= \lim_{h\to 0} \frac{60(h)}{h(\sqrt{4+h} + 2)} = \lim_{h\to 0} \frac{60}{\sqrt{4+h} + 2} \overset{\text{DSP}}{=} \frac{60}{\sqrt{4} + 2} = \frac{60}{4} = \boxed{15 \text{ km/hr}}$$