Math 111-01, Fall 2023 Professor Rob Benedetto

Supplement to Worksheet 2

This handout presents some methods for approaching limit problems like those on the second page of Worksheet 2, as well as examples that you should use as **model solutions** for those kinds of problems. A key tool we'll use is the following:

The Direct Substitution Property:

If $f(x)$ is a function that involves only $+, -, \cdot, \div$, and $\sqrt[n]{}$, then

$$\boxed{\lim_{x\to a} f(x) = f(a)} \text{ **IF** the resulting value } f(a) \text{ makes sense.}$$

That is, for functions with nice formulas involving only familiar ingredients like $+, -, \cdot, \div$, and $\sqrt[n]{}$, we can compute $\lim\limits_{x\to a} f(x)$ by just plugging in $x = a$ (also known as substituting $x = a$), provided that doesn't lead to an illegal move like dividing by zero or taking the square root of a negative number.

You can abbreviate the direct substitution property by DSP if you like. To show this in your solutions, you can show your work in the style of examples on the rest of this handout:

Example 1. $\displaystyle \lim_{x\to -3} \frac{x^2-4x-8}{x^2+2x-7} \overset{\text{DSP}}{=} \frac{9-4(-3)-8}{9+2(-3)-7} = \frac{9+12-8}{9-6-7} = \frac{13}{-4} = \boxed{-\frac{13}{4}}$

However, it is **NOT ALLOWED** to use DSP if the result would be undefined (because of dividing by zero, taking the square root of a negative number, etc.):

Example 2. $\displaystyle \lim_{x\to 3} \frac{x^2-4x+3}{x^2-9} \overset{\text{DSP}}{=} \frac{9-12+3}{9-9} = \frac{0}{0}$ **ILLEGAL MOVE!** No DSP allowed!!!

To fix this, sometimes algebra will help (especially if you got 0/0). So let's try that same problem again:

Example 3. $\displaystyle \lim_{x\to 3} \frac{x^2-4x+3}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x-1)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{(x-1)}{(x+3)} \overset{\text{DSP}}{=} \frac{2}{6} = \boxed{\frac{1}{3}}$

LOOK CLOSELY at Example 3 again. Notice that:

- the $\lim\limits_{x\to 3}$ sign **continues to appear** right up until the moment that we used DSP.
- I used = signs. In general, you should **always** follow this simple rule:
  - when you mean that two things are equal, use =
  - when you mean *anything else*, then **do not** use =.

Note that Example 3 resolved a 0/0 case. What if only ONE of the numerator or denominator is 0?

Example 4. $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-4} \overset{\text{DSP}}{=} \frac{0}{5} = \boxed{0}$

[0/5 is **not** illegal; it's just zero!]

Example 5. $\lim_{x\to 3} \frac{x^2-4x+1}{x^2-9}$ is $\frac{-2}{0}$, so we check both sides:

LHL: $\lim_{x\to 3^-} \frac{x^2-4x+1}{x^2-9} = \frac{-2}{0^-} = +\infty$

RHL: $\lim_{x\to 3^+} \frac{x^2-4x+1}{x^2-9} = \frac{-2}{0^+} = -\infty$

Since LHL$\neq$RHL, the original limit $\boxed{\lim_{x\to 3} \frac{x^2-4x+1}{x^2-9} \text{ DNE}}$

(Or if you prefer, you can say, "The original limit diverges.")