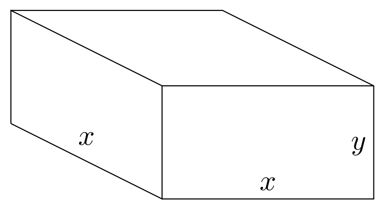
Solutions to Practice Test A for Midterm Exam 3

1. **(20 points)** You need to construct a box with a square base with a fixed volume of 24 cubic feet. The material for the bottom and top costs \$3 per square foot, and the material for the sides costs \$1 per square foot. What are the **dimensions** that minimize the cost required to build such a box? What is that **minimum cost**?

Solution. Here's the diagram:



The area of the base is x^2, so the base costs $\$3x^2$.
Similarly, the top also costs $\$3x^2$. So the base and top together cost $\$6x^2$.
Each of the four sides has area xy, and so costs $\$xy$. Together, the four sides cost $\$4xy$.

So the **total cost of the box (in dollars)** is $6x^2 + 4xy$

The **volume** of the box is x^2y, which we set equal to 24. Solving for y gives $y = 24x^{-2}$.

So the cost of the box is $C(x) = 6x^2 + 96x^{-1}$.

The common sense bounds say $x > 0$ [note that $x = 0$ is impossible because $x^2y = 24$]. [And $y > 0$ then gives $24/x^2 > 0$, which tells us nothing new.]

So we must minimize $C(x) = 6x^2 + 96x^{-1}$ on the domain $(0, \infty)$.

Differentiate: $C'(x) = 12x - 96x^{-2}$, which is **defined everywhere** on $(0, \infty)$.

Solving $C' = 0$ gives $12x^3 = 96$, so $x^3 = 8$, so $x = 2$ is the only critical number.
Using FDTAE, since $C'(x) = 12x^{-2}(x^3 - 8)$, our C' chart is:

x	$(0, 2)$	$(2, \infty)$
$C'(x)$	$-$	$+$
$C(x)$	$\searrow$	$\nearrow$

So by FDTAE, C has an absolute minimum at $x = 2\,\text{ft}$.

That gives $y = 24/2^2 = 6\,\text{ft}$ and $C(2) = 6(4) + 96/2 = \$72$.

So the best box is $\boxed{2\,\text{ft} \times 2\,\text{ft} \times 3\,\text{ft}}$ and $\boxed{\text{costs } \$72}$

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2. **(25 points)** Compute $\displaystyle\int_1^3 x^2 - 3x\, dx$ using each of the following **two** different methods:

(a) The Fundamental Theorem of Calculus.

(b) Riemann Sums and the limit definition of the definite integral.

Solution. (a): Let $f(x) = x^2 - 3x$, and chop the interval $[1, 3]$ into n equal-length intervals.

We have $\Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}$, and so $x_i = 1 + \dfrac{2i}{n}$.

The n-th right-hand Riemann sum is therefore

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right) \cdot \frac{2}{n} = \sum_{i=1}^{n} \left[\left(1 + \frac{2i}{n}\right)^2 - 3\left(1 + \frac{2i}{n}\right)\right]\frac{2}{n}$$

$$= \sum_{i=1}^{n} \left[1 + \frac{4i}{n} + \frac{4i^2}{n^2} - 3 - \frac{6i}{n} \right] \frac{2}{n} = \sum_{i=1}^{n} \left[-2 - \frac{2i}{n} + \frac{4i^2}{n^2} \right] \frac{2}{n}$$

$$= -\frac{4}{n} \sum_{i=1}^{n} 1 - \frac{4}{n^2} \sum_{i=1}^{n} i + \frac{8}{n^3} \sum_{i=1}^{n} i^2 = -\frac{4}{n} \cdot n - \frac{4}{n^2} \cdot \frac{n(n+1)}{2} + \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= -4 - 2\left(1 + \frac{1}{n}\right) + \frac{4}{3}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

Thus, $\int_1^3 x^2 - 3x\,dx = \lim_{n\to\infty} R_n = -4 - 2(1) + \frac{4}{3}(1)(2) = -6 + \frac{8}{3} = \boxed{-\frac{10}{3}}$

(b): $\int_1^3 x^2 - 3x\,dx = \frac{1}{3}x^3 - \frac{3}{2}x^2 \Big|_1^3 = \left(\frac{1}{3} \cdot 3^3 - \frac{3}{2} \cdot 3^2\right) - \left(\frac{1}{3} \cdot 1^3 - \frac{3}{2} \cdot 1^2\right)$

$= 9 - \frac{27}{2} - \frac{1}{3} + \frac{3}{2} = 9 - 12 - \frac{1}{3} = -3 - \frac{1}{3} = \boxed{-\frac{10}{3}}$

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3. **(30 points)** Let $f(x) = \dfrac{-x^2 + x + 2}{x^2 - 2x + 1} = \dfrac{-x^2 + x + 2}{(x-1)^2}$.

For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve $y = f(x)$.

You may take my word for it that:

$$f'(x) = \frac{x-5}{(x-1)^3} \qquad \text{and} \qquad f''(x) = \frac{-2x+14}{(x-1)^4}$$

Solution.

• Domain: $f(x)$ has domain $\{x | x \neq 1\}$

• VA: Vertical asymptote at $x = 1$ (because denom$= 0$ there, and nowhere else).

• HA: Horizontal asymptote is $y = -1$ on both sides since $\lim_{x\to\pm\infty} f(x) = -1$ because

$$\lim_{x\to\pm\infty} \frac{-x^2 + x + 2}{x^2 - 2x + 1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{-1 + \frac{1}{x} + \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}} = -1$$

• First Derivative Information:

We know $f'(x) = \dfrac{x-5}{(x-1)^3}$. The critical points occur where f' is undefined or zero. The former happens when $x = 1$, but $x = 1$ was not in the domain of the original function, so it isn't technically a critical number. The latter happens when $x = 5$. As a result, $x = 5$ is the critical number. Our chart for f' is:

x	$(-\infty, 1)$	$(1, 5)$	$(5, \infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

So f is $\boxed{\text{decreasing on } (1,5)}$ and $\boxed{\text{increasing on } (-\infty,1) \text{ and } (5,\infty)}$ with a $\boxed{\text{local minimum at } x = 5}$

(Remember that $x = 1$ is a vertical asymptote, not a max.)

• Second Derivative Information:

Meanwhile, $f'' = \dfrac{-2x+14}{(x-1)^4}$, which is **always defined** away from the asymptote $x = 1$.

We have $f'' = 0$ when $x = 7$. Thus, our f'' chart is

x	$(-\infty, 1)$	$(1, 7)$	$(7, \infty)$
$f''(x)$	$+$	$+$	$-$
$f(x)$	$\cup$	$\cup$	$\cap$

So f is $\boxed{\text{concave down on } (7, \infty))}$ and $\boxed{\text{concave up on } (-\infty, 1) \text{ and } (1, 7)}$ with $\boxed{\text{an inflection point at } x = 7}$

[For graph, see separate PDF.]

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4. **(10 points)** Compute the following limits.

(a) $\displaystyle\lim_{x \to \infty} \frac{x^2 - x + 5}{3x^7 + x^6 - 2022}$ (b) $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{4x^2 + 5x}}{6x + 7}$

Solution. (a) $\displaystyle\lim_{x \to \infty} \frac{x^2 - x + 5}{3x^7 + x^6 - 2022} \cdot \frac{\frac{1}{x^7}}{\frac{1}{x^7}} = \lim_{x \to \infty} \frac{x^{-5} - x^{-6} + 5x^{-7}}{3 + x^{-1} - 2022x^{-7}} = \frac{0}{3} = \boxed{0}$

(b) $\displaystyle\lim_{x \to -\infty} \frac{\sqrt{4x^2 + 5x}}{6x + 7} \cdot \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to -\infty} \frac{\sqrt{4x^2 + 5x} \cdot \left(-\sqrt{x^{-2}}\right)}{6 + 7x^{-1}} = \lim_{x \to -\infty} \frac{-\sqrt{4 + 5x^{-1}}}{6 + 7x^{-1}} = \frac{-\sqrt{4}}{6} = \boxed{-\frac{1}{3}}$

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5. **(15 points)** Compute the following definite and indefinite integrals.

(a) $\displaystyle\int \frac{y^2 - 4y + 1}{\sqrt{y}}\, dy$ (b) $\displaystyle\int_{-\pi/4}^{5\pi/6} 3x + \cos x\, dx$

Solution. (a) $\displaystyle\int \frac{y^2 - 4y + 1}{\sqrt{y}}\, dy = \int y^{3/2} - 4y^{1/2} + y^{-1/2}\, dy = \boxed{\frac{2}{5}y^{5/2} - \frac{8}{3}y^{3/2} + 2y^{1/2} + C}$

(b) $\displaystyle\int_{-\pi/4}^{5\pi/6} 3x + \cos x\, dx = \frac{3}{2}x^2 + \sin x \Big|_{-\pi/4}^{5\pi/6} = \left(\frac{3}{2}\left(\frac{5\pi}{6}\right)^2 + \sin\frac{5\pi}{6}\right) - \left(\frac{3}{2}\left(-\frac{\pi}{4}\right)^2 + \sin\left(-\frac{\pi}{4}\right)\right)$

$\displaystyle = \frac{3\dot{5}^2\pi^2}{2 \cdot 6^2} + \frac{1}{2} - \frac{3\pi^2}{2 \cdot 4^2} - \left(-\frac{\sqrt{2}}{2}\right) = \boxed{\left(\frac{25}{24} - \frac{3}{32}\right)\pi^2 + \frac{1 + \sqrt{2}}{2}}$