

Solutions to Practice Test B for Midterm Exam 2

1. (25 points) Compute the following derivatives by any legal method. Simplify when asked to do so.

(a). Compute $f'(x)$, where $f(x) = x^2 \sec(3x - 5)$. Simplify your answer.

(b). Compute $g''(x)$ (the *second* derivative), where $g(x) = \frac{x^2 - 4x + 3}{\sqrt{x}}$. Simplify your answer.

(c). Compute $h(x) = \sqrt{x} + \frac{1}{\sqrt{x}} + \frac{1}{1 + \sqrt{x}} + \frac{1}{\sqrt{1+x}}$. (Do not simplify your answer.)

(d). Compute $g'(x)$, where $g(x) = \left(\frac{1}{x^3} + \pi\right)^4 \cdot \left(x^4 - \frac{1}{x^7}\right)^{-5}$ (Do not simplify your answer.)

Solutions. (a) $f'(x) = 2x \sec(3x - 5) + x^2 \sec(3x - 5) \tan(3x - 5) \cdot 3 = \boxed{x \sec(3x - 5) [2 + 3x \tan(3x - 5)]}$

(b) $g(x) = x^{3/2} - 4x^{1/2} + 3x^{-1/2}$, so $g'(x) = \frac{3}{2}x^{1/2} - 2x^{-1/2} - \frac{3}{2}x^{-3/2}$,

and hence $\boxed{g''(x) = \frac{3}{4}x^{-1/2} + x^{-3/2} + \frac{9}{4}x^{-5/2}}$

(c) $h(x) = x^{1/2} + x^{-1/2} + (1 + x^{1/2})^{-1} + (1 + x)^{-1/2}$, so
 $h'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} - (1 + x^{1/2})^{-2} \cdot \frac{1}{2}x^{-1/2} - \frac{1}{2}(1 + x)^{-3/2}$

(d) We have $g(x) = \left(x^{-3} + \pi\right)^4 \left(x^4 - x^{-7}\right)^{-5}$, so

$\boxed{g'(x) = 4\left(x^{-3} + \pi\right)^3 \cdot (-3x^{-4}) \cdot \left(x^4 - x^{-7}\right)^{-5} + \left(x^{-3} + \pi\right)^4 \cdot (-5)\left(x^4 - x^{-7}\right)^{-6} \cdot (4x^3 + 7x^{-8})}$

2. (15 points) Find the **locations** of the **absolute maximum** and **absolute minimum value(s)** of the function

$$F(x) = (x - 1)^2(2x - 10)^2 \quad \text{on the interval} \quad [0, 4].$$

Solutions. (By Closed Interval Method):

$$\begin{aligned} F'(x) &= 2(x - 1) \cdot 1(2x - 10)^2 + (x - 1)^2 \cdot 2(2x - 10) \cdot 2 = 2(x - 1)(2x - 10)[(2x - 10) + 2(x - 1)] \\ &= 2(x - 1)(2x - 10)[4x - 12] = 16(x - 1)(x - 5)(x - 3), \end{aligned}$$

which is **always defined**.

Solving $F' = 0$ gives $x = 1, 3, 5$, but 5 is not in the interval.

So the only critical numbers are $x = 1, 3$. Testing them and endpoints:

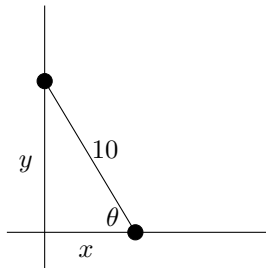
$$F(0) = (-1)^2(-10)^2 = 100, \quad F(1) = 0, \quad F(3) = 2^2(-4)^2 = 64, \quad F(4) = 3^2(-2)^2 = 36.$$

The maximum value is 100, and the minimum value is 0.

So the **absolute max occurs at $x = 0$** and the **absolute min occurs at $x = 1$**

3. (18 points) The top of a 10 foot ladder is sliding down a vertical wall at the rate of 1 foot per second. Consider the angle formed by the bottom of the ladder and the ground. How fast is this angle changing when the top of the ladder is 6 feet above the ground?

Solution. Here's the **Picture:**



Variables:

x = distance from bottom of ladder to wall, in ft

y = distance from top of ladder to floor, in ft

θ = angle of ladder with floor, in radians

(And t = time, in sec)

Main **Equation:** $\sin \theta = \frac{y}{10}$, i.e., $10 \sin \theta = y$

Differentiate (implicitly, w.r.t. time): $10 \cos \theta \frac{d\theta}{dt} = \frac{dy}{dt}$

Use key moment info:

At the key moment, we have $y = 6$ ft and we are told that $\frac{dy}{dt} = -1$ ft/sec

Also, when $y = 6$, we have $x = 8$, because $x^2 + y^2 = 10^2$, so $x^2 + 36 = 100$, i.e. $x^2 = 64$.

Thus, at the key moment, we have $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{10} = \frac{8}{10}$.

Plugging these values into the derivative equation above,

we have $10 \cdot \left(\frac{8}{10}\right) \frac{d\theta}{dt} = -1$, so $\frac{d\theta}{dt} = -\frac{1}{8}$.

That is, the angle is decreasing at $\frac{1}{8}$ rad/sec

4. **(14 points)** Consider the equation: $xy^3 + y \cos x = 3 + y^2 \sin x$
Find the equation of the tangent line to this curve at the point $(0, 3)$.

Solution. Implicit Diff: $y^3 + 3xy^2 \frac{dy}{dx} + \frac{dy}{dx} \cos x - y \sin x = 2y \frac{dy}{dx} \sin x + y^2 \cos x$,

so at the point $(x, y) = (0, 3)$, we have $3^3 + 0 + \frac{dy}{dx} \cos(0) - 3 \sin(0) = 2(3) \frac{dy}{dx} \sin(0) + 3^2 \cos(0)$,

i.e., $27 + \frac{dy}{dx} = 9$, i.e., $\frac{dy}{dx} = -18$ is the slope of the tangent line.

So the line is $y - 3 = -18(x - 0)$, i.e., $y = -18x + 3$

5. **(12 points)** Suppose $f(x)$ is a function with the property that

$$f(2) = 5, \quad f'(2) = -1, \quad f(4) = 3, \quad \text{and} \quad f'(4) = 2.$$

Let $g(x) = f(x^2)$ and $h(x) = (f(x))^2$. Compute $g'(2)$ and $h'(2)$.

Solution. $g'(x) = f'(x^2) \cdot 2x$, so $g'(2) = f'(4) \cdot (4) = (2)(4) = \boxed{8}$

$h'(2) = 2(f(2)) \cdot f'(2) = 2(5)(-1) = \boxed{-10}$