Solutions to Practice Test B for Midterm Exam 1

1. [32 Points] Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 0}\frac{x^2-4x}{x^2-3x-18}=$ **(b)** $\displaystyle\lim_{x\to -7}\frac{x^2+2x-35}{|x+7|}=$

(c) $\displaystyle\lim_{x\to -3}\frac{x^2+2x-3}{x^2-4x-12}=$ **(d)** $\displaystyle\lim_{x\to -6}\frac{\frac{x}{x+2}-\frac{x-3}{x}}{x+6}=$

(e) $\displaystyle\lim_{x\to 2}\frac{x+7}{x-2}=$ **(f)** $\displaystyle\lim_{x\to 3}\frac{g(x^2)-7}{(g(x))^2-1}=$ where $g(x)=x-2$

Solutions. (a): $\displaystyle\lim_{x\to 0}\frac{x^2-4x}{x^2-3x-18}\overset{\text{DSP}}{=}\boxed{0}$

(b): $\displaystyle\lim_{x\to -7}\frac{x^2+2x-35}{|x+7|}$ is piecewise defined at $x=-7$, so we check both sides:

$$\text{LHL}=\lim_{x\to -7^-}\frac{x^2+2x-35}{|x+7|}=\lim_{x\to -7^-}\frac{(x+7)(x-5)}{-(x+7)}=\lim_{x\to -7^-}-(x-5)\overset{\text{DSP}}{=}12$$

$$\text{RHL}=\lim_{x\to -7^+}\frac{x^2+2x-35}{|x+7|}=\lim_{x\to -7^+}\frac{(x+7)(x-5)}{(x+7)}=\lim_{x\to -7^+}x-5\overset{\text{DSP}}{=}-12$$

LHL$\neq$RHL, so $\boxed{\text{the original limit DNE}}$

(c): $\displaystyle\lim_{x\to -3}\frac{x^2+2x-3}{x^2-4x-12}\overset{\text{DSP}}{=}\frac{9-6-3}{9+12-12}=\frac{0}{9}=\boxed{0}$

(d): $$\lim_{x\to -6}\frac{\frac{x}{x+2}-\frac{x-3}{x}}{x+6}=\lim_{x\to -6}\frac{\frac{x}{x+2}-\frac{x-3}{x}}{x+6}\cdot\left(\frac{x(x+2)}{x(x+2)}\right)=\lim_{x\to -6}\frac{x^2-(x+2)(x-3)}{x(x+2)(x+6)}$$

$$=\lim_{x\to -6}\frac{x^2-(x^2-x-6)}{x(x+2)(x+6)}=\lim_{x\to -6}\frac{x+6}{x(x+2)(x+6)}=\lim_{x\to -6}\frac{1}{x(x+2)}\overset{\text{DSP}}{=}\frac{1}{(-6)(-4)}=\boxed{\frac{1}{24}}$$

(e): $\displaystyle\lim_{x\to 2}\frac{x+7}{x-2}$ is $\frac{9}{0}$, so we check both sides:

$$\text{LHL}=\lim_{x\to 2^-}\frac{x+7}{x-2}=\frac{9}{0^-}=-\infty \qquad \text{RHL}=\lim_{x\to 2^+}\frac{x+7}{x-2}=\frac{}{0^-}=+\infty$$

LHL$\neq$RHL, so $\boxed{\text{the original limit DNE}}$

(f) $g(x)=x-2$ gives $\displaystyle\lim_{x\to 3}\frac{g(x^2)-7}{(g(x))^2-1}=\lim_{x\to 3}\frac{x^2-2-7}{(x-2)^2-1}$

$$=\lim_{x\to 3}\frac{x^2-9}{x^2-4x+3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)(x-1)}=\lim_{x\to 3}\frac{x+3}{x-1}\overset{\text{DSP}}{=}\frac{6}{2}=\boxed{3}$$

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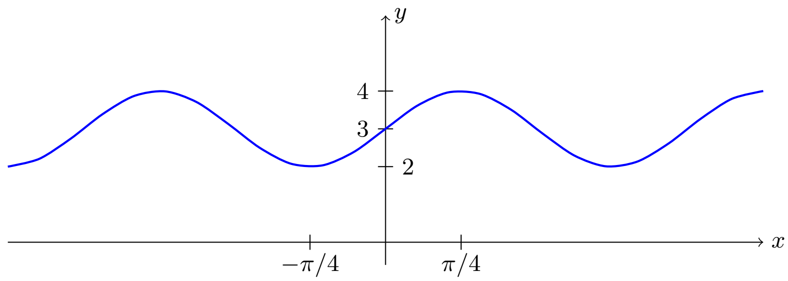
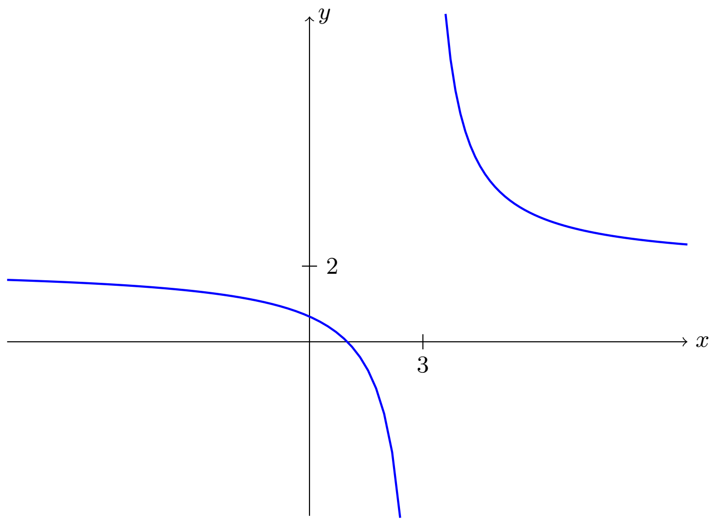
2. [12 Points] Use translation, etc. to graph the following two functions:

$$f(x)=3+\sin(2x) \qquad g(x)=2+\frac{1}{x-3}$$

Solution. (a): This is $y=\sin x$ translated **up by 3** and **compressed horizontally by 2**. So it looks like:



(b): This is $y = 1/x$ translated **right by 3** and **up by 2** So it looks like:



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3. [16 Points] Suppose that $f(x) = \sqrt{3 - x + x^2}$. Compute $f'(x)$ using the **limit definition of the derivative**.

Solution. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{3-(x+h)+(x+h)^2} - \sqrt{3-x+x^2}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{3-(x+h)+(x+h)^2} - \sqrt{3-x+x^2}}{h} \cdot \left(\frac{\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}}{\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}}\right)$$

$$= \lim_{h\to 0} \frac{3-(x+h)+(x+h)^2 - (3-x+x^2)}{h(\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2})}$$

$$= \lim_{h\to 0} \frac{3-x-h+x^2+2xh+h^2-3+x-x^2}{h(\sqrt{3-(x+h)} + \sqrt{3-x})} = \lim_{h\to 0} \frac{-h-2xh-h^2}{h(\sqrt{3-(x+h)} + \sqrt{3-x})}$$

$$= \lim_{h\to 0} \frac{h(-1+2x+h)}{h(\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2})} = \lim_{h\to 0} \frac{-1+2x+h}{\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}}$$

$$\overset{\text{DSP}}{=} \frac{-1+2x}{\sqrt{3-x+x^2} + \sqrt{3-x+x^2}} = \boxed{\frac{-1+2x}{2\sqrt{3-x+x^2}}}$$

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4. [14 Points] Suppose that $f(x) = x^3 + 7x^2 - 4x + 9$. Write the **equation of the tangent line** to the curve $y = f(x)$ when $x = -1$. **Use the limit definition of the derivative when computing the derivative.**

Solution. We compute $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{((x+h)^3 + 7(x+h)^2 - 4(x+h) + 9) - (x^3 + 7x^2 - 4x + 9)}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 7x^2 + 14xh + 7h^2 - 4x - 4h + 9 - x^3 - 7x^2 + 4x - 9}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 + 14xh + 7h^2 - 4h}{h} = \lim_{h\to 0} \frac{h(3x^2 + 3xh + h^2 + 14x + 7h - 4)}{h}$$

$$= \lim_{h\to 0} 3x^2 + 3xh + h^2 + 14x + 7h - 4 = \boxed{3x^2 + 14x - 4}$$

Thus, the slope of the tangent line is $f'(-1) = 3 - 14 - 4 = -15$.

The point is given by $(-1, f(-1)) = (1, 19)$. So the tangent line is

$$y - 19 = -15(x - (-1)), \quad \text{i.e.,} \qquad \boxed{y = -15x + 4}$$

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5. [10 Points] Suppose that f and g are functions, and

- $\lim_{x\to 7} f(x) = 4$
- $\lim_{x\to 7} g(x) = -2$
- $g(x)$ is continuous at $x = 7$.

Evaluate the following quantities and fully justify your answers. Do not just put down a value:

(a) $\lim_{x\to 7}(3f(x) - 5g(x)) =$ **(b)** $g(7) =$

Solutions. (a): By the limit laws,

$\lim_{x\to 7}(3f(x) - 5g(x)) = 3\lim_{x\to 7} f(x) - 5\lim_{x\to 7} g(x) = 3(4) - 5(-2) = \boxed{22}$

(b): By definition of continuity, $g(7) = \lim_{x\to 7} g(x) = \boxed{-2}$

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6. (16 Points) Consider the function defined by

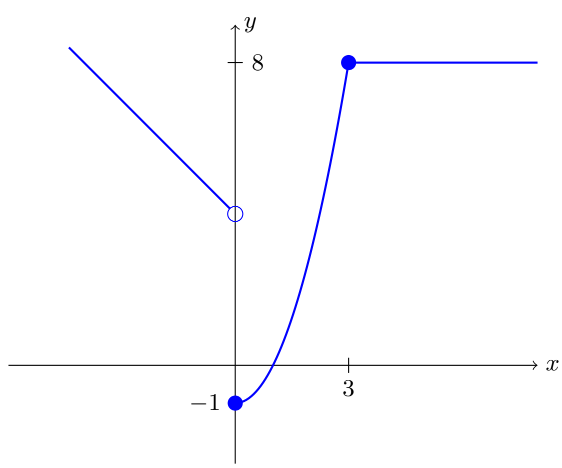
$$f(x) = \begin{cases} 4 - x & \text{if } x < 0 \\ x^2 - 1 & \text{if } 0 \le x \le 3 \\ 8 & \text{if } x > 3 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.

(b) Compute $\lim_{x\to 0} f(x)$. **(c)** Compute $\lim_{x\to 3} f(x)$.

(d) State the value(s) at which f is discontinuous. Justify your answers using definitions or theorems discussed in class.

Solutions. (a) Using translation/scaling for the various pieces, and then putting them together, here's the graph:



(b): Checking both sides:

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} 4 - x \overset{\text{DSP}}{=} 4$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x^2 - 1 \overset{\text{DSP}}{=} -1$

So $\lim\limits_{x\to 0} f(x)$ $\boxed{\text{DNE}}$ because RHL≠LHL

(c): Checking both sides:

$\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} x^2 - 1 \overset{\text{DSP}}{=} 8$

$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} 8 \overset{\text{DSP}}{=} 8$

So $\boxed{\lim\limits_{x\to 3} f(x) = 8}$ because RHL=LHL= 8

(d): f is discontinuous at $x = 0.$, because $\lim\limits_{x\to 0} f(x)$ DOES NOT EXIST.

But everywhere else, f is continuous, even at $x = 3$, because even at $x = 3$, we have $\lim\limits_{x\to 3} f(x) = f(3)$.