Solutions to Practice Test A for Midterm Exam 1

1. [30 Points] Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(**a**) $\displaystyle\lim_{x\to-3} \frac{x^2-2x-15}{x^2+x-6} =$ (**b**) $\displaystyle\lim_{x\to5} \frac{x^2-2x-15}{|5-x|} =$

(**c**) $\displaystyle\lim_{x\to2} \frac{x^2-2x-15}{x^2+x-6} =$ (**d**) $\displaystyle\lim_{x\to5} \frac{x^2-2x-15}{x^2+x-6} =$

(**e**) $\displaystyle\lim_{x\to2} \frac{x+7}{(x-2)^2} =$ (**f**) $\displaystyle\lim_{x\to-1} \frac{H(x+1)-H(-1-x)}{x+1} =$ where $H(x)=\sqrt{x+2}$

Solutions. (a): $\displaystyle\lim_{x\to-3} \frac{x^2-2x-15}{x^2+x-6} = \lim_{x\to-3} \frac{(x+3)(x-5)}{(x+3)(x-2)} = \lim_{x\to-3} \frac{x-5}{x-2} \overset{\text{DSP}}{=} \frac{-8}{-5} = \boxed{\frac{8}{5}}$

(b): $\displaystyle\lim_{x\to5} \frac{x^2-2x-15}{|5-x|}$ is piecewise defined right at $x=5$, so we check both sides:

$$\text{LHL} = \lim_{x\to5^-} \frac{x^2-2x-15}{|5-x|} = \lim_{x\to5^-} \frac{(x+3)(x-5)}{-(x-5)} = \lim_{x\to5^-} -(x+3) \overset{\text{DSP}}{=} -8$$

$$\text{RHL} = \lim_{x\to5^+} \frac{x^2-2x-15}{|5-x|} = \lim_{x\to5^+} \frac{(x+3)(x-5)}{(x-5)} = \lim_{x\to5^+} (x+3) \overset{\text{DSP}}{=} 8$$

LHL≠RHL, so $\boxed{\text{the original limit DNE}}$

(c): $\displaystyle\lim_{x\to2} \frac{x^2-2x-15}{x^2+x-6} = \lim_{x\to2} \frac{(x+3)(x-5)}{(x+3)(x-2)} = \lim_{x\to2} \frac{x-5}{x-2}$ is $\frac{-3}{0}$, so we check both sides:

$$\text{LHL} = \lim_{x\to2^-} \frac{x-5}{x-2} = \frac{-3}{0^-} = +\infty \qquad \text{RHL} = \lim_{x\to2^+} \frac{x-5}{x-2} = \frac{-3}{0^+} = -\infty$$

LHL≠RHL, so $\boxed{\text{the original limit DNE}}$

(d): $\displaystyle\lim_{x\to5} \frac{x^2-2x-15}{x^2+x-6} \overset{\text{DSP}}{=} \frac{25-10-15}{25+5-6} = \frac{0}{24} = \boxed{0}$

(e): $\displaystyle\lim_{x\to2} \frac{x+7}{(x-2)^2}$ is $\frac{7}{0}$, so we check both sides:

$$\text{LHL} = \lim_{x\to2^-} \frac{x+7}{(x-2)^2} = \frac{9}{(0^-)^2} = +\infty \qquad \text{RHL} = \lim_{x\to2^+} \frac{x+7}{(x-2)^2} = \frac{9}{(0^+)^2} = +\infty$$

LHL=RHL, so $\boxed{\text{the original limit diverges to } +\infty}$

(f): $\displaystyle\lim_{x\to-1} \frac{H(x+1)-H(-1-x)}{x+1} = \lim_{x\to-1} \frac{\sqrt{(x+1)+2}-\sqrt{(-1-x)+2}}{x+1}$

$$= \lim_{x\to-1} \frac{\sqrt{(x+1)+2}-\sqrt{(-1-x)+2}}{x+1} \cdot \left(\frac{\sqrt{(x+1)+2}+\sqrt{(-1-x)+2}}{\sqrt{(x+1)+2}+\sqrt{(-1-x)+2}}\right)$$

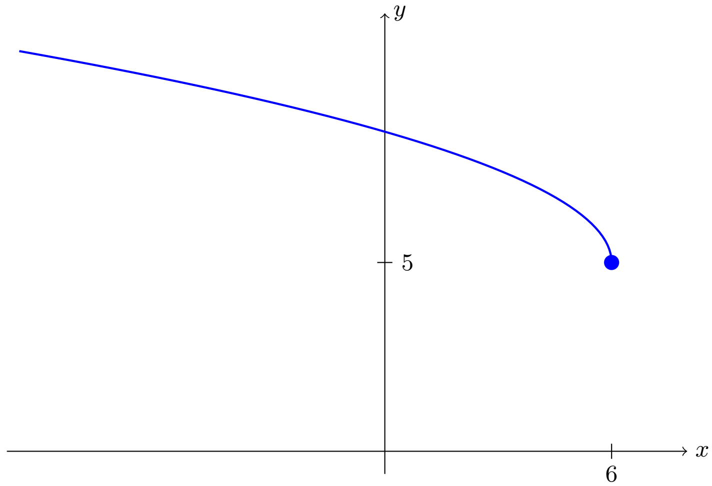
$$= \lim_{x\to-1} \frac{(x+3)-(1-x)}{(x+1)(\sqrt{x+3}+\sqrt{1-x})} = \lim_{x\to-1} \frac{2x+2}{(x+1)(\sqrt{x+3}+\sqrt{1-x})}$$

$$= \lim_{x\to-1} \frac{2(x+1)}{(x+1)(\sqrt{x+3}+\sqrt{1-x})} = \lim_{x\to-1} \frac{2}{\sqrt{x+3}+\sqrt{1-x}} \overset{\text{DSP}}{=} \frac{2}{\sqrt{2}+\sqrt{2}} = \boxed{\frac{1}{\sqrt{2}}}$$

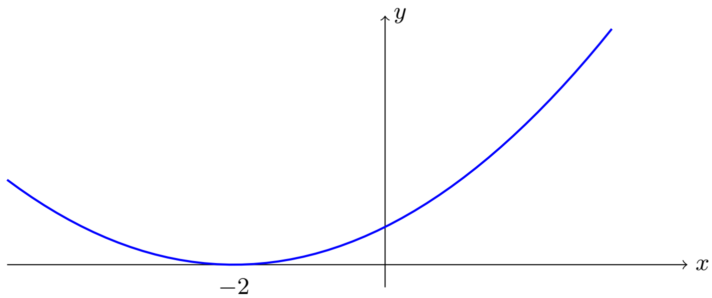
2. [13 Points] Use translation, etc. to graph the following two functions:

$$f(x) = 5 + \sqrt{6-x} \qquad\qquad g(x) = \frac{1}{10}(x+2)^2$$

Solutions. (a): This is $y = \sqrt{x}$ translated **left by 6** to make $y = \sqrt{x+6}$ and *then* **reflected left-to-right** to make $y = \sqrt{6-x}$, and finally translated **up by 5**. So it looks like:



(b): This is $y = x^2$ translated **left by 2** and then **compressed vertically by 10**. So it looks like:



3. [15 Points] Suppose that $f(x) = \dfrac{x+7}{x-3}$. Compute $f'(x)$ using the **limit definition of the derivative**.

Solution. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{(x+h)+7}{(x+h)-3} - \dfrac{x+7}{x-3}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{x+h+7}{x+h-3} - \dfrac{x+7}{x-3}}{h} \cdot \left(\frac{(x+h-3)(x-3)}{(x+h-3)(x-3)}\right) = \lim_{h\to 0} \frac{(x+h+7)(x-3)-(x+7)(x+h-3)}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{x^2+xh+7x-3x-3h-21-(x^2+xh-3x+7x+7h-21)}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{x^2+xh+7x-3x-3h-21-x^2-xh+3x-7x-7h+21}{h(x+h-3)(x-3)} = \lim_{h\to 0} \frac{-10h)}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{-10}{(x+h-3)(x-3)} \overset{\text{DSP}}{=} \frac{-10}{(x-3)(x-3)} = \boxed{\frac{-10}{(x-3)^2}}$$

4. [10 Points] Suppose that $f(x) = x^2 - 7x - 12$. Write the **equation of the tangent line** to the curve $y = f(x)$ when $x = -2$. **Use the limit definition of the derivative when computing the derivative.**

Solution. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{(x+h)^2 - 7(x+h) - 12 - (x^2 - 7x - 12)}{h}$

$= \lim_{h\to 0} \dfrac{x^2 + 2xh + h^2 - 7x - 7h - 12 - x^2 + 7x + 12}{h} = \lim_{h\to 0} \dfrac{2xh + h^2 - 7h}{h}$

$= \lim_{h\to 0} 2x + h - 7 \overset{\text{DSP}}{=} 2x - 7.$

Thus, the slope of the tangent line is $f'(-2) = 2(-2) - 7 = -11$.

We also have $f(-2) = (-2)^2 - 7(-2) - 12 = 4 + 14 - 12 = 6$, so the point is $(-2, 6)$. Therefore, the tangent line is

$$y - 6 = -11(x - (-2)), \quad \text{i.e.,} \qquad \boxed{y = -11x - 16}$$

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5. [12 Points] Suppose that f and g are functions, **and**

- $\lim_{x\to 7} f(x) = 5$
- $\lim_{x\to 7} g(x) = -3$
- $f(5) = 7$
- $g(x)$ is continuous at $x = 7$.

Evaluate the following quantities and fully **justify** your answers. Do not just put down a value:

(a) $\lim_{x\to 7} \sqrt{3f(x) - 7g(x)} =$

(b) $\lim_{x\to 7} \dfrac{f(x)}{1-x} =$

(c) $g(7) =$

(d) $g \circ f(5) =$

Solutions. (a): By the limit laws, $\lim_{x\to 7} \sqrt{3f(x) - 7g(x)} = \sqrt{\lim_{x\to 7} 3f(x) - 7g(x)}$

$= \sqrt{3\lim_{x\to 7} f(x) - 7\lim_{x\to 7} g(x)} = \sqrt{3(5) - 7(-3)} = \sqrt{15 + 21} = \sqrt{36} = \boxed{6}$

(b): By the limit laws, $\lim_{x\to 7} \dfrac{f(x)}{1-x} = \dfrac{\lim_{x\to 7} f(x)}{\lim_{x\to 7} 1 - x} = \dfrac{5}{1-7} = \boxed{-\dfrac{5}{6}}$

(c): By definition of continuity, $g(7) = \lim_{x\to 7} g(x) = \boxed{-3}$

(d): By part (c) and the assumptions, $g \circ f(5) = g(f(5)) = g(7) = \boxed{-3}$

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6. [20 Points] Consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-3} & \text{if } x > 3 \\ 1 & \text{if } x = 3 \\ 6 - 2x & \text{if } 0 < x < 3 \\ 16 - x^2 & \text{if } -4 < x \le 0 \\ \dfrac{1}{x+4} & \text{if } x < -4 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.

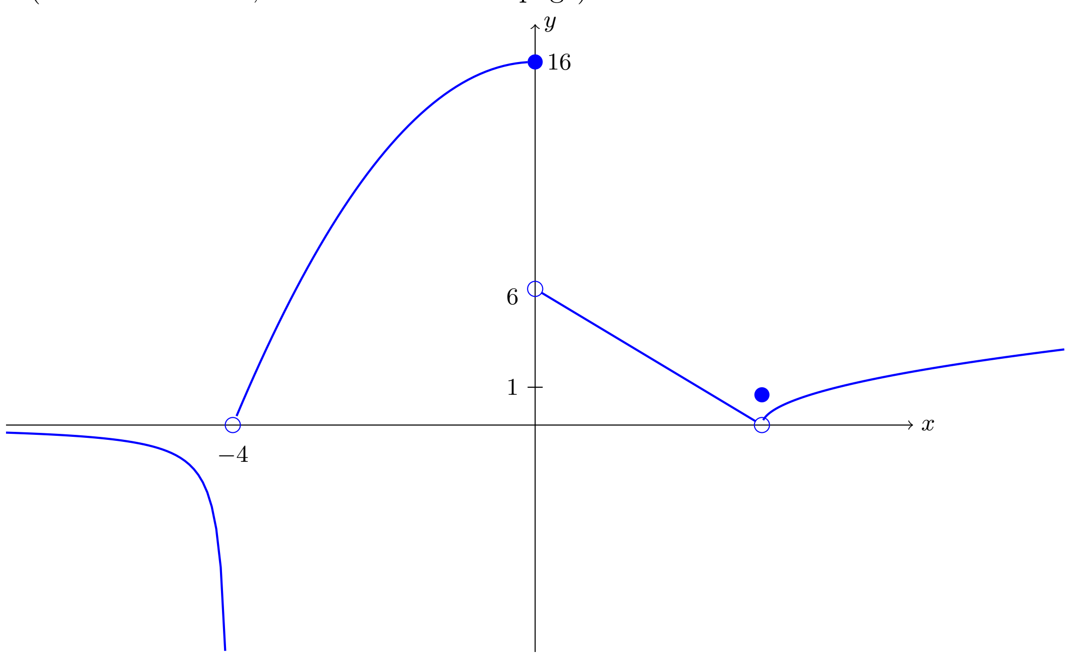
(b) State the **Domain** of the function $f(x)$.

(c) Compute $\begin{cases} \lim_{x\to 0^+} f(x) = \\ \lim_{x\to 0^-} f(x) = \\ \lim_{x\to 0} f(x) = \end{cases}$ **(d)** Compute $\begin{cases} \lim_{x\to 3^+} f(x) = \\ \lim_{x\to 3^-} f(x) = \\ \lim_{x\to 3} f(x) = \end{cases}$

(e) Compute $\begin{cases} \lim_{x\to -4^+} f(x) = \\ \lim_{x\to -4^-} f(x) = \\ \lim_{x\to -4} f(x) = \end{cases}$

(f) State the value(s) at which f is discontinuous. Justify your answer(s) using definitions or theorems discussed in class.

Solutions. (a) Using translation/scaling for the various pieces, and then putting them together, here's the graph (axes not to scale, to fit better on the page)



(b): each of the functions is defined on the portion assigned to it, but $x = -4$ is not in any of the portions. So $\boxed{\text{the domain is all real numbers except } -4}$, or if you prefer, $\{x | x \neq -4\}$

(c): $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 6 - 2x \overset{\text{DSP}}{=} \boxed{6}$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 16 - x^2 \overset{\text{DSP}}{=} \boxed{16}$

$\lim_{x\to 0} f(x)$ $\boxed{\text{DNE}}$ because RHL≠LHL

(d): $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x-3} \overset{\text{DSP}}{=} \boxed{0}$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 6 - 2x \overset{\text{DSP}}{=} \boxed{0}$

$\lim_{x\to 3} f(x) = \boxed{0}$ because RHL=LHL$= 0$

(e): $\lim_{x\to -4^+} f(x) = \lim_{x\to -4^+} 16 - x^2 \overset{\text{DSP}}{=} 16 - 16 = \boxed{0}$

$\lim_{x\to -4^-} f(x) = \lim_{x\to -4^-} \frac{1}{x+4} = \frac{1}{0-} = \boxed{-\infty}$

$\lim_{x\to -4} f(x)$ $\boxed{\text{DNE}}$ because RHL$\neq$LHL

(f): f is discontinuous at $x = -4, 0, 3$.

• f is discontinuous at $x = 3$, because despite the fact that $f(3) = 1$ is defined, and $\lim_{x\to 3} f(x) = 0$, those two values are not equal.

• f is discontinuous at $x = 0$, because despite the fact that $f(0) = 16$ is defined, the $\lim_{x\to 0} f(x)$ DOES NOT EXIST.

• f is discontinuous at $x = -4$ for two reasons, $f(-4)$ is undefined, and the $\lim_{x\to -4} f(x)$ DOES NOT EXIST.

• But everywhere else, f is continuous.