Math 111-01, Fall 2022 Professor Rob Benedetto

Solutions to Midterm Exam 2

1. **(16 points; 2 parts)** Compute the following derivatives by any legal method. Simplify your answers.

(a) $f'(x)$, where $f(x) = x^3 \tan(4 - 5x)$

(b) The second derivative $M''(x)$, where $M(x) = \dfrac{x^3 - 4x + 6}{\sqrt{x}}$

Solutions. (a): $f'(x) = 3x^2 \tan(4-5x) + x^3 \sec^2(4-5x) \cdot (-5) = \boxed{3x^2 \tan(4-5x) - 5x^3 \sec^2(4-5x)}$

(b): $M(x) = x^{5/2} - 4x^{1/2} + 6x^{-1/2}$, so $M'(x) = \frac{5}{2}x^{3/2} - 2x^{-1/2} - 3x^{-3/2}$, and hence

$$\boxed{M'(x) = \frac{15}{4}x^{1/2} + x^{-3/2} + \frac{9}{2}x^{-5/2}}$$

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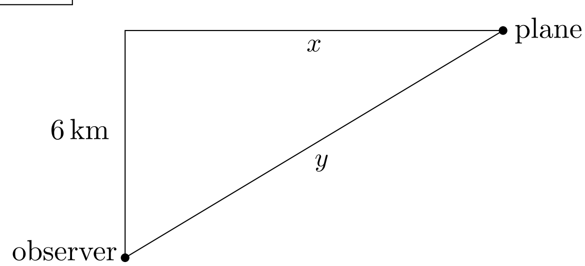
2. **(20 points)** A jet plane, flying in a straight line at a constant altitude of 6 km, passes directly over an observer on the ground. When the jet has flown a further 8 km (i.e., 8 km past the point on its path that was directly above the observer), its speed is $\frac{1}{3}$ km/sec. At that moment, how fast is the (diagonal) distance from the observer to the jet increasing?

Solution. Here's the **Picture**:



Variables:
x =horizontal distance plane is from the spot directly above observer, in km
y =diagonal distance, in km

(And t =time, in sec)

Main **Equation**: $y^2 = 6^2 + x^2$

Differentiate (implicitly, w.r.t. time): $2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$

Use key moment info:

At the key moment, we have $x = 8$ km and we are told that $\dfrac{dx}{dt} = \dfrac{1}{3}$km/sec

Also, plugging $x = 8$ into the original equation gives $y^2 = 36 + 64 = 100$, so $y = \pm 100$. Given the picture, we have $y = 10\,\text{km}$

Plugging $x = 8$, $y = 10$ and $\dfrac{dx}{dt} = \dfrac{1}{3}$ into the derivative equation above,

we have $2(8)\dfrac{1}{3} = 2(10)\dfrac{dy}{dt}$ i.e., $\dfrac{dy}{dt} = \boxed{\dfrac{4}{15}\text{ km/sec}}$

3. **(20 points)** Let $F(x) = \dfrac{x+4}{x^2+9}$.

Find the absolute minimum and absolute maximum values of F on the interval $[-4, 4]$.

Solution. (By Closed Interval Method):

$F'(x) = \dfrac{1(x^2+9)-(x+4)(2x)}{(x^2+9)^2} = \dfrac{x^2+9-2x^2-8x}{(x^2+9)^2} = \dfrac{-x^2-8x+9}{(x^2+9)^2} = \dfrac{-(x+9)(x-1)}{(x^2+9)^2}$,

which is **always defined**.

Solving $F' = 0$ gives $x = -9, 1$, but -9 is not in the interval.

So the only critical number is $x = 1$. Testing it and endpoints:

$$F(-4) = \frac{0}{16+9} = 0, \quad F(1) = \frac{5}{1+9} = \frac{1}{2}, \quad F(4) = \frac{8}{16+9} = \frac{8}{25}$$

So the $\boxed{\text{absolute maximum is } \frac{1}{2}}$ and the $\boxed{\text{absolute minimum is } 0}$

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4. **(15 points)** Consider the equation: $x^4 + 2y^4 = x^3y + xy^3 + 5$

Find the equation of the tangent line to this curve at the point $(1, -1)$.

Solution. By Implicit Diff: $4x^3 + 8y^3y' = 3x^2y + x^3y' + y^3 + 3xy^2y'$,

So at $(x, y) = (1, -1)$, we have $4 - 8y' = -3 + y' - 1 + 3y'$.

That is, $4 - 8y' = -4 + 4y'$, which becomes $12y' = 8$, so $y' = \dfrac{2}{3}$.

Thus, by point-slope, the tangent line is $y + 1 = \dfrac{2}{3}(x-1)$, i.e., $y + 1 = \dfrac{2}{3}x - \dfrac{5}{3}$, so $\boxed{y = \dfrac{2}{3}x - \dfrac{5}{3}}$

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5. **(15 points)** Suppose $f(x)$ is a function with the property that

$$f(1) = -1, \qquad f'(1) = 2, \qquad f(5) = -6, \qquad \text{and} \qquad f'(5) = 7.$$

Let $G(x) = f(x^3 + 4x^2)$, and let $H(x) = [f(x)]^3 + 4[f(x)]^2$. Compute $G'(1)$ and $H'(1)$.

Solution. We have $G'(x) = f'(x^3+4x^2) \cdot (3x^2+8x)$, so $G'(1) = f'(5) \cdot (3+8) = 7 \cdot 11 = \boxed{77}$

We have $H'(x) = 3f(x)^2 f'(x) + 8f(x)f'(x)$,
so $H'(1) = 3f(1)^2 f'(1) + 8f(1)f'(1) = 3(-1)^2 \cdot 2 + 8(-1) \cdot 2 = 6 - 16 = \boxed{-10}$

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6. **(14 points)** Let $H(x) = \sin^2(4x) + \sec(3x) - 5\cos(2x)$. Compute $H'\left(\dfrac{\pi}{12}\right)$. Simplify your answer.

Solution. We have $H'(x) = 2\sin(4x)\cos(4x) \cdot 4 + \sec(3x)\tan(3x) \cdot 3 - 5(-\sin(2x)) \cdot 2$,

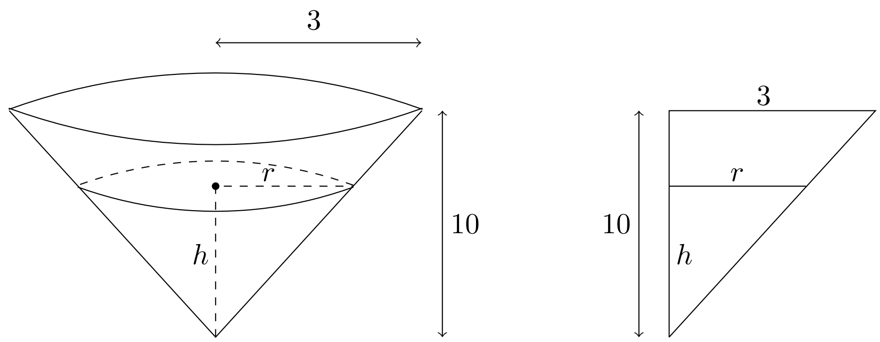
so $H'\left(\dfrac{\pi}{12}\right) = 8\sin\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{3}\right) + 3\sec\left(\dfrac{\pi}{4}\right)\tan\left(\dfrac{\pi}{4}\right) + 10\sin\left(\dfrac{\pi}{6}\right)$

$= 8 \cdot \dfrac{\sqrt{3}}{2} \cdot \dfrac{1}{2} + 3\dfrac{1}{\sqrt{2}/2} \cdot \dfrac{\sqrt{2}/2}{\sqrt{2}/2} + 10 \cdot \dfrac{1}{2}$

$= 2\sqrt{3} + 3 \cdot \dfrac{2}{\sqrt{2}} + 5 = \boxed{2\sqrt{3} + 3\sqrt{2} + 5}$

OPTIONAL BONUS A. (2 points.) A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). The cup is being filled with water, and at the moment when the depth of the water is 5 cm, the water level is rising at 1/2 cm/sec. How fast is water being poured into the cup at that moment?

Solution. **Picture**:



Variables:
r = radius of **water** in tank (cm)
h = depth of **water** in tank (cm)
V = volume of **water** in tank (cm^3)
[Key moment: $h = 5\text{cm}$ and $\dfrac{dh}{dt} = 1/2\text{cm/sec}$]

Equations: $V = \dfrac{\pi}{3} r^2 h$ and (by similar triangles) $\dfrac{h}{10} = \dfrac{r}{3}$, so $r = \dfrac{3h}{10}$, so $V = (\pi/3) r^2 h = (3\pi/100) h^3$.

Deriv: $\dfrac{dV}{dt} = \dfrac{3\pi}{100} \cdot 3h^2 \dfrac{dh}{dt}$.

Solve: At key moment, with $h = 5$ and $dh/dt = 1/2$: $\dfrac{dV}{dt} = \dfrac{9\pi \cdot 25}{200} = \boxed{\dfrac{9\pi}{8} \text{cm}^3/\text{sec}}$

OPTIONAL BONUS B. (1 point.) In the past week, the term of China's current head of state was renewed, and also in Britain, a new prime minister was appointed. Name both of these two political leaders.

Answer. The President of China is Xi Jinping, and the Prime Minister of Britain is Rishi Sunak.

Xi was renewed to a third term as General Secretary of the Chinese Communist Party on October 22.

Sunak was appointed by the Conservative Party on October 25, a few days after his predecessor Liz Truss resigned.