**Solutions to Midterm Exam 1**

1. **(34 points)** Evaluate each of the following limits. Please **justify** your answers. Be clear about whether the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

**(a)** $\displaystyle\lim_{x\to-2}\frac{x^2+7x+10}{x^2-2x-8}$ **(b)** $\displaystyle\lim_{x\to-7}\frac{x^2+2x-35}{|x+7|}$

**(c)** $\displaystyle\lim_{x\to3}\frac{x^2+x-12}{(x-3)^3}$ **(d)** $\displaystyle\lim_{x\to1}\frac{2-\sqrt{x+3}}{x^2-1}$

**Solutions**. (a): $\displaystyle\lim_{x\to-2}\frac{x^2+7x+10}{x^2-2x-8}=\lim_{x\to-2}\frac{(x+2)(x+5)}{(x+2)(x-4)}=\lim_{x\to-2}\frac{x+5}{x-4}\overset{\text{DSP}}{=}\frac{3}{-6}=\boxed{-\frac{1}{2}}$

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(b): Because $|x+7|$ is piecewise, we look at both sides:

LHL: $\displaystyle\lim_{x\to-7^-}\frac{x^2+2x-35}{|x+7|}=\lim_{x\to-7^-}\frac{(x+7)(x-5)}{-(x+7)}=\lim_{x\to-7^-}-(x-5)\overset{\text{DSP}}{=}-(-12)=12$

RHL: $\displaystyle\lim_{x\to-7^+}\frac{x^2+2x-35}{|x+7|}=\lim_{x\to-7^+}\frac{(x+7)(x-5)}{x+7}=\lim_{x\to-7^+}x-5\overset{\text{DSP}}{=}-12$

Since LHL$\neq$RHL, $\boxed{\text{the original limit DNE}}$

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(c): $\displaystyle\lim_{x\to3}\frac{x^2+x-12}{(x-3)^3}=\lim_{x\to3}\frac{(x-3)(x+4)}{(x-3)^3}=\lim_{x\to3}\frac{x+4}{(x-3)^2}$ which is $\frac{7}{0}$, so we check both sides:

LHL: $\displaystyle\lim_{x\to3^-}\frac{x+4}{(x-3)^2}=\frac{7}{(0^-)^2}=+\infty$

RHL: $\displaystyle\lim_{x\to3^+}\frac{x+4}{(x-3)^2}=\frac{7}{(0^+)^2}=+\infty$

Since LHL=RHL$=+\infty$, $\boxed{\text{the original limit diverges to }+\infty}$

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(d): $\displaystyle\lim_{x\to1}\frac{2-\sqrt{x+3}}{x^2-1}=\lim_{x\to1}\frac{2-\sqrt{x+3}}{x^2-1}\left(\frac{2+\sqrt{x+3}}{2+\sqrt{x+3}}\right)=\lim_{x\to1}\frac{4-(x+3)}{(x^2-1)(2+\sqrt{x+3})}$

$\displaystyle=\lim_{x\to1}\frac{-(x-1)}{(x-1)(x+1)(2+\sqrt{x+3})}=\lim_{x\to1}\frac{-1}{(x+1)(2+\sqrt{x+3})}\overset{\text{DSP}}{=}\frac{-1}{2(2+\sqrt{4})}=\boxed{-\frac{1}{8}}$

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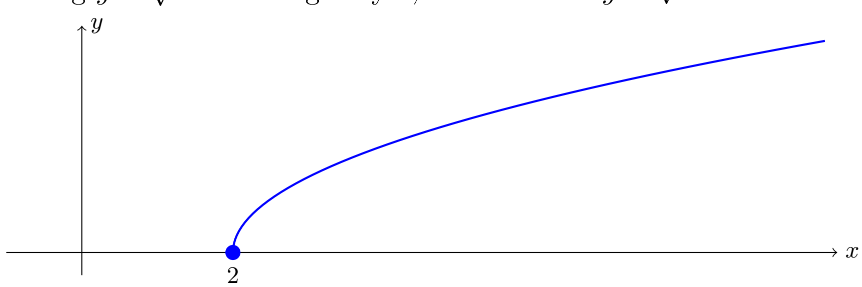
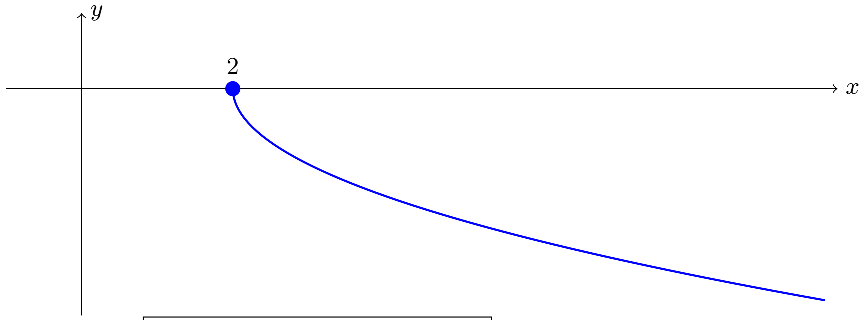
2. **(8 points)** Use translation, etc. to graph the function $f(x)=4-\sqrt{x-2}$

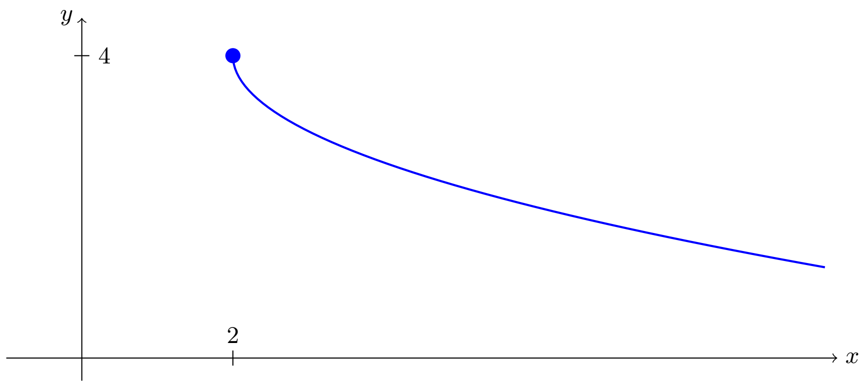
**Solution**. Translating $y=\sqrt{x}$ to the right by 2, we have that $y=\sqrt{x-2}$ looks like this:




Reflecting vertically, we have that $y=-\sqrt{x-2}$ looks like:




Translating up by 4, then, $\boxed{y = 4 - \sqrt{x-2} \text{ looks like:}}$

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3. **(16 points)** Let $f(x) = \dfrac{x+7}{x-3}$. Compute $f'(x)$ using the **limit definition of the derivative**.

**Solution**. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{(x+h)+7}{(x+h)-3} - \dfrac{x+7}{x-3}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{x+h+7}{x+h-3} - \dfrac{x+7}{x-3}}{h} \cdot \left(\frac{(x+h-3)(x-3)}{(x+h-3)(x-3)}\right) = \lim_{h\to 0} \frac{(x+h+7)(x-3)-(x+7)(x+h-3)}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{x^2+xh+7x-3x-3h-21-(x^2+xh-3x+7x+7h-21)}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{x^2+xh+7x-3x-3h-21-x^2-xh+3x-7x-7h+21}{h(x+h-3)(x-3)} = \lim_{h\to 0} \frac{-10h}{h(x+h-3)(x-3)}$$

$$= \lim_{h\to 0} \frac{-10}{(x+h-3)(x-3)} \overset{\text{DSP}}{=} \frac{-10}{(x-3)(x-3)} = \boxed{\frac{-10}{(x-3)^2}}$$

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4. **(14 points)** Let $g(x) = x^2 - 7x - 4$. Find an equation of the tangent line to the curve $y = g(x)$ at the point where $x = -1$.

Use the **limit definition of the derivative** when differentiating.

**Solution**. $g'(x) = \lim\limits_{h\to 0} \dfrac{g(x+h)-g(x)}{h} = \lim\limits_{h\to 0} \dfrac{(x+h)^2 - 7(x+h) - 4 - (x^2-7x-4)}{h}$

$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-7x-7h-4-x^2+7x+4}{h} = \lim_{h\to 0} \frac{2xh+h^2-7h}{h}$$

$$= \lim_{h\to 0} 2x+h-7 \overset{\text{DSP}}{=} 2x-7.$$

Thus, the slope of the tangent line is $g'(-1) = 2(-1) - 7 = -9$.

We also have $g(-2) = (-1)^2 - 7(-1) - 4 = 1 + 7 - 4 = 4$, so the point is $(-1, 4)$. Therefore, the tangent line is

$$y - 4 = -9(x - (-1)), \quad \text{i.e.,} \qquad \boxed{y = -9x - 5}$$

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5. **(10 points)** Let $f$ and $g$ be functions such that

$$\lim_{x\to 4} f(x) = -2 \qquad \lim_{x\to 4} g(x) = 3 \qquad g(-2) = 4 \qquad f \text{ is continuous at } x = 4$$

Evaluate the following, and **justify your answers**. (Justifications can be brief, but they can't be completely absent.)

**(a)** $\lim_{x\to 4}\left(3f(x) + \left(g(x)\right)^2\right)$ **(b)** $f(4)$ **(c)** $f \circ g(-2)$

**Solutions**. (a): By the limit laws, $\lim_{x\to 4}\left(3f(x)+\left(g(x)\right)^2\right) = 3\lim_{x\to 4} f(x)+\left(\lim_{x\to 4} g(x)\right)^2 = 3(-2)+(3)^2 = -6+9 = \boxed{3}$

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(b): By definition of continuity, $f(4) = \lim_{x\to 4} f(x) = \boxed{-2}$

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(c): By part (b), $f \circ g(-2) = f(g(-2)) = f(4) = \boxed{-2}$

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6. **(18 points)** Let $f(x) = \begin{cases} \dfrac{1}{x-1} & \text{if } x < 2, \\ (x-3)^2 & \text{if } x \geq 2. \end{cases}$

**(a)** What is the domain of $f$? (And briefly justify, of course.)

**(b)** Compute: $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, and $\lim_{x\to 2} f(x)$

**(c)** Carefully sketch the graph $y = f(x)$.

**(d)** State the value(s) at which $f$ is discontinuous. Justify your answer(s) briefly.

**Solutions**. (a): The two portions ($x < 2$ and $x \geq 2$) cover the whole real line, but $1/(x-1)$ is undefined at $x = 1$, which is in its portion. So **the domain is** $\{x | x \neq 1\}$

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(b): $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{x-1} \overset{\text{DSP}}{=} \frac{1}{1} = \boxed{1}$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x-3)^2 \overset{\text{DSP}}{=} (-1)^2 = \boxed{1}$

Since RHL=LHL= 1, the original limit is $\boxed{\lim_{x\to 2} f(x) = 1}$

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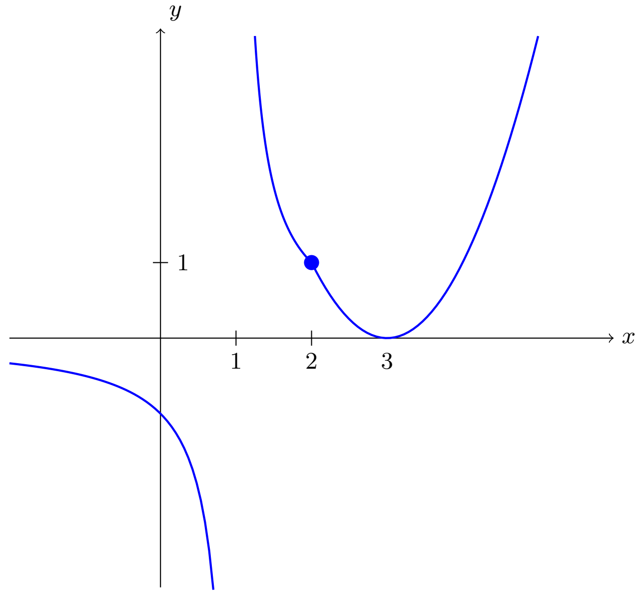
(c): Using translation/scaling for the various pieces, and then putting them together, here's the graph:

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(d): $f$ is discontinuous at $x = 1$ because $f(1)$ is undefined (as noted in part (a)). The only other possible problem point is the piecewise point $x = 2$, but in (b) we saw that $\lim_{x\to 2} f(x) = 1 = f(2)$, so $f$ is continuous at $x = 2$.

So $\boxed{\text{the only point of discontinuity is } x = 1}$

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**OPTIONAL BONUS A. (2 points.)** Let $g(x) = \sqrt{x^3 - 4x^2 - 7}$. Compute $g'(x)$ using the **limit definition of the derivative**.

**Solution**. $g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^3 - 4(x+h)^2 - 7} - \sqrt{x^3 - 4x^2 - 7}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^3 - 4(x+h)^2 - 7} - \sqrt{x^3 - 4x^2 - 7}}{h} \left( \frac{\sqrt{(x+h)^3 - 4(x+h)^2 - 7} + \sqrt{x^3 - 4x^2 - 7}}{\sqrt{(x+h)^3 - 4(x+h)^2 - 7} + \sqrt{x^3 - 4x^2 - 7}} \right)$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - 4(x+h)^2 - 7 - (x^3 - 4x^2 - 7)}{h(\sqrt{(x+h)^3 - 4(x+h)^2 - 7} + \sqrt{x^3 - 4x^2 - 7})}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 4x^2 - 8xh - 4h^2 - 7 - x^3 + 4x^2 + 7}{h(\sqrt{(x+h)^3 - 4(x+h)^2 - 7} + \sqrt{x^3 - 4x^2 - 7})}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 8xh - 4h^2}{h(\sqrt{(x+h)^3 - 4(x+h)^2 - 7} + \sqrt{x^3 - 4x^2 - 7})}$$

$$= \lim_{h\to 0} \frac{3x^2 + 3xh + h^2 - 8x - 4h}{\sqrt{(x+h)^3 - 4(x+h)^2 - 7} + \sqrt{x^3 - 4x^2 - 7}} \overset{\text{DSP}}{=} \frac{3x^2 - 8x}{\sqrt{x^3 - 4x^2 - 7} + \sqrt{x^3 - 4x^2 - 7}}$$

$$= \boxed{\frac{3x^2 - 8x}{2\sqrt{x^3 - 4x^2 - 7}}}$$

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**OPTIONAL BONUS B. (1 point.)** There are five countries that are permanent members of the United Nations Security Council. Name them.

**Answer**. China, France, Russia, UK, USA