A SOLUTION TO EXERCISE 8.16 OF DYNAMICS IN ONE NON-ARCHIMEDEAN VARIABLE

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Abstract. Exercise 8.16 of my Dynamics in One Non-Archimedean Variable book asks for a proof of Theorem 8.15(f), that the (Berkovich space) boundary of the filled Julia set of a polynomial coincides with its Julia set. The proof is pretty hard, though, so here’s a sketch.

Let \( \phi \in \mathbb{C}_{v}(z) \) be a rational function of degree at least 2. The Berkovich filled Julia set of \( \phi \) is

\[
K_{\phi,an} := \{ \zeta \in \mathbb{P}^1_{an} : \lim_{n \to \infty} \phi^n(\zeta) \neq \infty \}.
\]

Theorem 8.15(f) of Dynamics in One Non-Archimedean Variable states the following:

**Theorem 8.15(f).** Let \( \phi \in \mathbb{C}_{v}(z) \) be a rational function of degree at least 2, with Berkovich Julia set \( J_{\phi,an} \) and Berkovich filled Julia set \( K_{\phi,an} \). Prove that \( J_{\phi,an} = \partial K_{\phi,an} \).

In the book I punt the proof to Exercise 8.16, saying the proof is “slightly different” from the type I analog that appears as Proposition 5.27. But that’s quite misleading; the proof is significantly harder than that of Proposition 5.27. So here is a sketch of a proof, using ideas that appear the proof of Theorem 9.5, which includes a proof of a more general statement about attracting Fatou components.

**Proof.** As in Proposition 5.27, there is some \( R > 0 \) so that if we set \( V_0 := \mathbb{P}^1_{an} \setminus \overline{D}_{an}(0,R) \), then \( \phi(V_0) \subseteq V_0 \), with \( \phi^n(\xi) \to \infty \) for all \( \xi \in V_0 \).

For each \( n \geq 1 \), define \( V_n := \phi^{-n}(V_0) \), which is a single connected open affinoid, since all \( d^n \) preimages of \( \infty \) are \( \infty \) itself. Since \( \phi(V_0) \subseteq V_0 \), we have \( V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \). Thus,

\[
K_{\phi,an} = \mathbb{P}^1_{an} \setminus V, \quad \text{where} \quad V := \bigcup_{n \geq 0} V_n.
\]

In particular, \( V \) is an open set (and is easily seen to be connected), \( K_{\phi,an} \) is closed, and they have the same boundary \( \partial V = \partial K_{\phi,an} \).

To show \( J_{\phi,an} = \partial K_{\phi,an} \), the inclusion (\( \subseteq \)) is easy, as follows. We have \( V_0 \subseteq \mathcal{F}_{\phi,an} \) since \( \phi(V_0) \subseteq V_0 \) and \( V_0 \) is open. Therefore, by Proposition 8.2(b), we have \( V \subseteq \mathcal{F}_{\phi,an} \); taking complements yields \( \partial K_{\phi,an} \supseteq J_{\phi,an} \).

Next, a short Lemma:

**Lemma.** Let \( \xi \in \partial K_{\phi,an} \).

1. If \( \xi' \in \mathbb{P}^1_{an} \setminus \{ \xi \} \) lies between \( \xi \) and \( \infty \), then \( \xi \in V \).
2. \( \phi(\xi) \in \partial K_{\phi,an} \).

(The proof of the Lemma is quick and left to reader.)
The rest of the proof of the Theorem is devoted to proving that $\partial K_{\phi,an} \subseteq J_{\phi,an}$. Given a point $\zeta \in \partial K_{\phi,an}$ and an open set $W$ containing $\zeta$, we must show $\bigcup_{n \geq 0} W$ omits only finitely many points of $\mathbb{P}^1_{an}$.

If $\zeta$ is of type II or III, let $C_0$ be the closed disk corresponding to $\zeta$. Then there is a slightly larger open disk $W_0$ such that the annulus $W' := W_0 \setminus C_0$ is contained in $W$.

Otherwise, i.e. if $\zeta$ is a point of type I or IV, then there is an open disk $W_0$ with $\zeta \in W_0 \subseteq W$. Let $W' := W_0$ in this case.

In either case, let $\zeta'$ be the unique boundary point of the disk $W_0$. For each $n \geq 1$, define $W_n := \phi^n(W_0)$ and (in the type II or III case) $C_n := \phi^n(C_0)$.

Since each $W_n$ is an open disk, $\phi$ is a polynomial, and $\zeta \in W_0$, a quick induction shows that for every $n \geq 0$, we have:

- $\phi^n(\zeta) \in W_n$;
- in the type II and III case, $C_n \subseteq W_n$, with $\partial C_n = \{\phi^n(\zeta)\}$;
- $\phi^n(W')$ is either $W_n$ (with one boundary point $\phi^n(\zeta')$) or else $W_n \setminus C_n$ (with two boundary points, $\phi^n(\zeta)$ and $\phi^n(\zeta')$).

The disk $W_0$ is an open neighborhood of $\zeta \in \partial K_{\phi,an}$, and hence it contains points of $V$, which approach $\infty$ under iteration. On the other hand, every $W_n$ contains $\phi^n(\zeta) \in K_{\phi,an} \subseteq \overline{D}_{an}(0,R)$. Thus, there is some $N \geq 0$ such that for every $n \geq N$, we have $W_n \supseteq \overline{D}_{an}(0,R)$. We consider three cases.

**Case 1.** There exists $m \geq N$ such that $\phi^m(W') = W_m$. (This case includes the case that $\xi$ is of type I or IV.) Since the boundary points $\phi^n(\zeta')$ approach $\infty$, it follows that

$$\bigcup_{n \geq 0} \phi^n(W) \supseteq \bigcup_{n \geq m} \phi^n(W') = \bigcup_{n \geq m} W_n = \mathbb{A}^1_{an},$$

and hence $W$ is not dynamically stable, as desired.

**Case 2:** We are not in Case 1, but there exist $\ell > m \geq N$ such that $\phi^\ell(\zeta) \neq \phi^m(\zeta)$. By the Lemma, neither of $\phi^\ell(\zeta)$ nor $\phi^m(\zeta)$ lies between the other and $\infty$, and hence the two closed sets $C_\ell$ and $C_m$ are disjoint. It follows that $\phi^\ell(W') \cup \phi^m(W') = W_\ell$.

Therefore, as in Case 1, we have

$$\bigcup_{n \geq 0} \phi^n(W) \supseteq \bigcup_{n \geq m} \phi^n(W') = \bigcup_{n \geq \ell} W_n = \mathbb{A}^1_{an},$$

and again $W$ is not dynamically stable.

**Case 3:** Finally, assume neither Case 1 nor Case 2 arises. Thus, $\xi$ is of type II or III, and for all $n \geq N$, we have $\phi^n(\zeta) = \phi^N(\zeta)$, and $\phi^n(W') = W_n \setminus C_N$. Then $\xi := \phi^N(\zeta)$ is a fixed point of $\phi$.

Let $u$ be the direction at $\xi$ towards $\infty$, and let $m := \deg_{\xi,u}(\phi)$ be the local degree of $\phi$ in that direction. Then there is an open disk disk $D_{an}(b,t)$ containing $\xi$ (and hence $C_N$) small enough that $\phi$ has Weierstrass degree $m$ on the open set $U := D_{an}(b,t) \setminus C_N$ extending from $\xi$ in the direction $u$. If $m = 1$, then since $\phi$ is a polynomial fixing $\xi$, we have $\phi(U) = U$, and hence $\phi(D_{an}(b,t)) = D_{an}(b,t)$. Therefore all points of $D_{an}(b,t)$ remain bounded under iteration of $\phi$, contradicting the fact that $\xi \in \partial K_{\phi,an}$.

By this contradiction, we must have $m \geq 2$. Thus, $\xi$ is a repelling fixed point; by Theorem 8.7, it is of type II (not that we need that here) and lies in $J_{\phi,an}$, as desired. □