A SOLUTION TO EXERCISE 8.16 OF *DYNAMICS IN ONE NON-ARCHIMEDEAN VARIABLE*

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Abstract. Exercise 8.16 of my *Dynamics in One Non-Archimedean Variable* book asks for a proof of Theorem 8.15(f), that the (Berkovich space) boundary of the filled Julia set of a polynomial coincides with its Julia set. The proof is pretty hard, though, so here's a sketch.

Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree at least 2. The *Berkovich filled Julia set* of ϕ is

$$\mathcal{K}_{\phi,\mathrm{an}} := \{\zeta \in \mathbb{P}^1_{\mathrm{an}} : \lim_{n\to\infty} \phi^n(\zeta) \neq \infty\}.$$

Theorem 8.15(f) of *Dynamics in One Non-Archimedean Variable* states the following:

Theorem 8.15(f). *Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree at least 2, with Berkovich Julia set $\mathcal{J}_{\phi,\mathrm{an}}$ and Berkovich filled Julia set $\mathcal{K}_{\phi,\mathrm{an}}$. Prove that $\mathcal{J}_{\phi,\mathrm{an}} = \partial\mathcal{K}_{\phi,\mathrm{an}}$.*

In the book I punt the proof to Exercise 8.16, saying the proof is "slightly different" from the type I analog that appears as Proposition 5.27. But that's quite misleading; the proof is significantly harder than that of Proposition 5.27. So here is a sketch of a proof, using ideas that appear the proof of Theorem 9.5, which includes a proof of a more general statement about attracting Fatou components.

Proof. As in Proposition 5.27, there is some $R > 0$ so that if we set $V_0 := \mathbb{P}^1_{\mathrm{an}} \smallsetminus \overline{D}_{\mathrm{an}}(0,R)$, then $\phi(V_0) \subseteq V_0$, with $\phi^n(\xi) \to \infty$ for all $\xi \in V_0$.

For each $n \geq 1$, define $V_n := \phi^{-n}(V_0)$, which is a single connected open affinoid, since all d^n preimages of ∞ are ∞ itself. Since $\phi(V_0) \subseteq V_0$, we have $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$. Thus,

$$\mathcal{K}_{\phi,\mathrm{an}} = \mathbb{P}^1_{\mathrm{an}} \smallsetminus V, \quad \text{where} \quad V := \bigcup_{n\geq 0} V_n.$$

In particular, V is an open set (and is easily seen to be connected), $\mathcal{K}_{\phi,\mathrm{an}}$ is closed, and they have the same boundary $\partial V = \partial\mathcal{K}_{\phi,\mathrm{an}}$.

To show $\mathcal{J}_{\phi,\mathrm{an}} = \partial\mathcal{K}_{\phi,\mathrm{an}}$, the inclusion $(\subseteq)$ is easy, as follows. We have $V_0 \subset \mathcal{F}_{\phi,\mathrm{an}}$ since $\phi(V_0) \subseteq V_0$ and V_0 is open. Therefore, by Proposition 8.2(b), we have $V \subseteq \mathcal{F}_{\phi,\mathrm{an}}$; taking complements yields $\partial\mathcal{K}_{\phi,\mathrm{an}} \supseteq \mathcal{J}_{\phi,\mathrm{an}}$.

Next, a short Lemma:

Lemma. Let $\xi \in \partial\mathcal{K}_{\phi,\mathrm{an}}$.

(1) If $\xi' \in \mathbb{P}^1_{\mathrm{an}} \smallsetminus \{\xi\}$ lies between ξ and ∞, then $\xi \in V$.
(2) $\phi(\xi) \in \partial\mathcal{K}_{\phi,\mathrm{an}}$.

(The proof of the Lemma is quick and left to reader.)

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The rest of the proof of the Theorem is devoted to proving that $\partial\mathcal{K}_{\phi,\mathrm{an}} \subseteq \mathcal{J}_{\phi,\mathrm{an}}$. Given a point $\zeta \in \partial\mathcal{K}_{\phi,\mathrm{an}}$ and an open set W containing ζ, we must show $\bigcup_{n\geq 0} W$ omits only finitely many points of $\mathbb{P}^1_{\mathrm{an}}$.

If ζ is of type II or III, let C_0 be the closed disk corresponding to ζ. Then there is a slightly larger open disk W_0 such that the annulus $W' := W_0 \smallsetminus C_0$ is contained in W.

Otherwise, i.e. if ζ is a point of type I or IV, then there is an open disk W_0 with $\xi \in W_0 \subseteq W$. Let $W' := W_0$ in this case.

In either case, let ζ' be the unique boundary point of the disk W_0. For each $n \geq 1$, define $W_n := \phi^n(W_0)$ and (in the type II or III case) $C_n := \phi^n(C_0)$.

Since each W_n is an open disk, ϕ is a polynomial, and $\zeta \in W_0$, a quick induction shows that for every $n \geq 0$, we have:

- $\phi^n(\zeta) \in W_n$,
- in the type II and III case, $C_n \subseteq W_n$, with $\partial C_n = \{\phi^n(\zeta)\}$,
- $\phi^n(W')$ is either W_n (with one boundary point $\phi^n(\zeta')$) or else $W_n \smallsetminus C_n$ (with two boundary points, $\phi^n(\zeta)$ and $\phi^n(\zeta')$).

The disk W_0 is an open neighborhood of $\zeta \in \partial\mathcal{K}_{\phi,\mathrm{an}}$, and hence it contains points of V, which approach ∞ under iteration. On the other hand, every W_n contains $\phi^n(\zeta) \in \mathcal{K}_{\phi,\mathrm{an}} \subseteq \overline{D}_{\mathrm{an}}(0,R)$. Thus, there is some $N \geq 0$ such that for every $n \geq N$, we have $W_n \supseteq \overline{D}_{\mathrm{an}}(0,R)$. We consider three cases.

Case 1. There exists $m \geq N$ such that $\phi^m(W') = W_m$. (This case includes the case that ξ is of type I or IV.) Since the boundary points $\phi^n(\zeta')$ approach ∞, it follows that

$$\bigcup_{n\geq 0} \phi^n(W) \supseteq \bigcup_{n\geq m} \phi^n(W') = \bigcup_{n\geq m} W_n = \mathbb{A}^1_{\mathrm{an}},$$

and hence W is not dynamically stable, as desired.

Case 2: We are not in Case 1, but there exist $\ell > m \geq N$ such that $\phi^\ell(\zeta) \neq \phi^m(\zeta)$. By the Lemma, neither of $\phi^\ell(\zeta)$ nor $\phi^m(\zeta)$ lies between the other and ∞, and hence the two closed sets C_ℓ and C_m are disjoint. It follows that $\phi^\ell(W') \cup \phi^m(W') = W_\ell$. Therefore, as in Case 1, we have

$$\bigcup_{n\geq 0} \phi^n(W) \supseteq \bigcup_{n\geq m} \phi^n(W') = \bigcup_{n\geq \ell} W_n = \mathbb{A}^1_{\mathrm{an}},$$

and again W is not dynamically stable.

Case 3: Finally, assume neither Case 1 nor Case 2 arises. Thus, ξ is of type II or III, and for all $n \geq N$, we have $\phi^n(\zeta) = \phi^N(\zeta)$, and $\phi^n(W') = W_n \smallsetminus C_N$. Then $\xi := \phi^N(\zeta)$ is a fixed point of ϕ.

Let $\boldsymbol{u}$ be the direction at ξ towards ∞, and let $m := \deg_{\xi,\boldsymbol{u}}(\phi)$ be the local degree of ϕ in that direction. Then there is an open disk disk $D_{\mathrm{an}}(b,t)$ containing ξ (and hence C_N) small enough that ϕ has Weierstrass degree m on the open set $U := D_{\mathrm{an}}(b,t) \smallsetminus C_N$ extending from ξ in the direction $\boldsymbol{u}$. If $m = 1$, then since ϕ is a polynomial fixing ξ, we have $\phi(U) = U$, and hence $\phi(D_{\mathrm{an}}(b,t)) = D_{\mathrm{an}}(b,t)$. Therefore all points of $D_{\mathrm{an}}(b,t)$ remain bounded under iteration of ϕ, contradicting the fact that $\xi \in \partial\mathcal{K}_{\phi,\mathrm{an}}$.

By this contradiction, we must have $m \geq 2$. Thus, ξ is a repelling fixed point; by Theorem 8.7, it is of type II (not that we need that here) and lies in $\mathcal{J}_{\phi,\mathrm{an}}$, as desired. □